Ramsey numbers

Ramsey numbers (formally defined shortly) are the outcome of work of the brilliant English mathematician-philosopher-economist Frank Ramsey\(^1\). In a seminal paper\(^2\) he proved the following result:

Let \(G\) be an infinite simple\(^3\) graph, then \(G\) has an infinite sub-graph \(G'\) every two of whose vertices are joined\(^4\), or an infinite sub-graph \(G'\) no two of whose vertices are joined\(^5\).

Erdös\(^6\) and Szekeres formulated a finite version of Ramsey’s result, known as the Erdös-Szekeres theorem. In crude simplistic terms, their theorem asserts that every finite simple graph, having at least a certain number of vertices, must contain either a clique of a certain size or an independent set of a certain size. More precisely – and only by way of an introductory example – their theorem asserts that if one chooses two numbers 7 and 8 (say), then every simple graph \(G\) having at least a certain number of vertices (\(N\), say) must contain either a clique of size 7, or an independent set of size 8. The smallest such ‘\(N\)’ for which that is true is known as the Ramsey number \(r(7, 8)\). Remarkably, \(r(7, 8)\) – and all other Ramsey numbers like it – does exist (as you will shortly see), but nobody\(^7\) knows its exact value!!

Most people encounter Ramsey number (perhaps without knowing so) through the well-known six-people-at-a-party-problem: prove that for any six people there must be at least three of them, every two of whom know each other, or three of them, no two of whom know each other (and – as a pointer to the significance of the ‘6’ in relation to the ‘3’ and ‘3’ – construct an example of ‘5’ people with no 3-clique and no 3-independent set. Care should be taken if introducing this problem to non-mathematicians, or, more precisely, acquaintances lacking the facility for abstract thought.)

In these notes I am summarizing work discussed at length in class and am resorting to graph theory language, with the obvious correspondences being understood:

- knowing \(\leftrightarrow\) joined
- not knowing \(\leftrightarrow\) not joined

(You ought to realise that resorting to such language is – for you and I – merely a convenience, and is not an integral part of what we are thinking about. To put it another way, should you ever try to introduce someone to Ramsey numbers, don’t start by rambling on about ‘graphs’, ‘vertices’, ‘edges’, ‘cliques’, ‘independent sets’, unless you wish to frighten everyone off.)

\(^1\) http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Ramsey.html
\(^3\) A graph is simple if every two of its vertices are either joined by a single edge, or are not joined at all.
\(^4\) Forming what is knows as an infinite clique.
\(^5\) Forming what is knows as an infinite independent set.
\(^6\) http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Erdos.html
\(^7\) Recall Erdös – talking about Ramsey numbers in the video \(N\) is a Number – remarking (in wonderment) that not even the value of \(r(5, 5)\) is known, though it is known to lie between 43 and 49. \(r(7, 8)\) is known to lie between 216 and 1031; quite a range!!
Solution (expressed in graph theory language) to the six-people-at-a-party-problem. Let $G$ be a simple graph with 6 vertices, and let $P$ be any one of those vertices. Since $P$ is joined to 5 or 4 or 3 or 2 or 1 or 0 of the other 5 vertices then

1. $P$ is joined to at least 3 of the other vertices, or
2. $P$ is not joined to at least 3 of the other vertices

In case (1), if some 2 of the other vertices ($P'$ and $P''$ say) are joined, then every pair from $P$, $P'$ and $P''$ are joined: they form a 3-clique (the people they represent are ‘mutually acquainted’); otherwise no 2 of the other 3 vertices ($P'$, $P''$, $P'''$ say) are joined: they form a 3-independent set (the people they represent are ‘mutually not acquainted’).

In case (2), if some 2 of the other vertices ($P'$ and $P''$ say) are not joined then no pair from $P$, $P'$ and $P''$ are joined: they form a 3-independent set (the people they represent are ‘mutually not acquainted’); otherwise every 2 of the other 3 vertices ($P'$, $P''$, $P'''$ say) are joined: they form a 3-clique (the people they represent are ‘mutually acquainted’).

**Definition.** Let $k, l \geq 2$, and suppose there is a number $N$ such that every simple graph with $N$ vertices has either a clique of size $k$ or an independent set of size $l$, then the minimum such $N$ is called the Ramsey number $r(k, l)$.

**Examples.**

1. A trivial one: $r(k, 2) = k$ for all $k \geq 2$. Why? It’s immediate: let $G$ be any simple graph having exactly $k$ vertices. If all $G$’s vertices are mutually joined then $G$ automatically has a $k$-clique (itself!), while if not all of $G$’s vertices are mutually joined then some 2 of $G$’s vertices are not joined, and so $G$ automatically has a 2-independent set.
   It is obvious that one may construct a simple graph with fewer than $k$ vertices, with no $k$-clique, and no 2-independent set. Thus $r(k, 2) = k$.

2. A trivial one: $r(2, l) = l$ for all $l \geq 2$. Why? It’s immediate: let $G$ be any simple graph having exactly $l$ vertices. If some 2 of $G$’s vertices are mutually joined then $G$ automatically has a 2-clique, while if no 2 of $G$’s vertices are joined then $G$ automatically has an $l$-independent set (itself!)
   It is obvious that one may construct a simple graph with fewer than $l$ vertices, with no 2-clique, and no $l$-independent set. Thus $r(2, l) = l$.

3. $r(3, 3) = 6$. Why? Well we already know that $r(3, 3)$ is at most 6, and all we have to do to show that it is actually 6 is to make up an example of a simple graph, having 5 vertices, which has no 3-clique and no 3-independent set. The obvious example is provided by the simple pentagon:

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8 Actually I have only introduced what might be called the 2-variable Ramsey numbers. There are several-variables generalizations (in which multi-colouring plays a part).

9 Such a graph is called a Ramsey graph. Not surprisingly they have lovely symmetry.
4. It should be obvious that \( r(k, l) = r(l, k) \). One should see that it is true. ‘Knowing’ and ‘not-knowing’, being ‘joined’ and ‘not being joined’, being joined by ‘red’ (say, to represent ‘knowing’) or ‘blue’ (say, to represent ‘not knowing’), \ldots, are, abstractly, the same.

**The Erdös-Szekeres theorem.** Suppose \( r(k - 1, l) \) and \( r(k, l - 1) \) both exist \((k, l \geq 2)\), then \( r(k, l) \) also exists and \( r(k, l) \leq r(k - 1, l) + r(k, l - 1) \).

**Proof.** Let \( G \) be a simple graph with \( r(k - 1, l) + r(k, l - 1) \) vertices\(^{10}\), and let \( P \) be any one of those vertices. Then, either

(a) \( P \) is joined to at least \( r(k - 1, l) \) of the other vertices\(^{11}\). Thus either some \((k - 1)\) of the \( r(k - 1, l) \) vertices form a \((k - 1)\)-clique – in which case \( P \), together with those \((k - 1)\) vertices, form a \( k \)-clique – or some \( l \) of those \( r(k - 1, l) \) vertices form an \( l \)-independent set. In either event \( G \) has a \( k \)-clique or an \( l \)-independent set.

or

(b) \( P \) is not joined to at least \( r(k, l - 1) \) of the other vertices\(^{12}\). Then either some \( k \) of the \( r(k, l - 1) \) vertices form a \( k \)-clique, or some \((l - 1)\) of those \( r(k, l - 1) \) vertices form an \((l - 1)\)-independent set, in which case \( P \), together with those \((l - 1)\) vertices, form an \( l \)-independent set. In either event \( G \) has a \( k \)-clique or an \( l \)-independent set.

That completes the proof.

**Comment.** One should recall from our discussions how absolutely critical it is to assert the correct minima in (a) and (b) (the ‘lift’ that I refer to in footnotes 11 and 12 should be your guiding principle). For consider the first non-trivial examples after \( r(3, 3) \), either \( r(4, 3) \) or \( r(3, 4) \). If one tries to argue (e.g.) that

\[
r(4, 3) \leq r(4 - 1, 3) + r(4, 3 - 1)
\]

namely

\[
r(4, 3) \leq 10
\]

\(^{10}\) In the \( r = 3 \) and \( l = 3 \) case the ‘6’ was the sum of \( r(2, 3) = 3 \) and \( r(3, 2) = 3 \).

\(^{11}\) The key idea now is to give a ‘lift’ – as it were – to that \((k - 1)\).

\(^{12}\) The key idea now is to give a ‘lift’ to that \((l - 1)\).
then one could attempt to proceed by saying of a vertex $P$ (in a 10 vertex simple graph) that it must be joined to 9 or 8 or … or 2 or 1 or 0 of the other 9 vertices. And one could then (correctly) assert that $P$ must be joined to a minimum of 5 vertices, or not joined to a minimum of 5 vertices. However, to do so, would lead one nowhere (in terms of arguing to the desired conclusion that $r(4, 3) \leq 10$).

Even if one were to (correctly) assert that $P$ must be joined to a minimum of 4 vertices, or not joined to a minimum of 6 vertices, that too would lead one nowhere (in terms of arguing to the desired conclusion that $r(4, 3) \leq 10$). Of course it would enable one to argue that $r(3, 4) \leq 10$ (note the ‘switch’ of the ‘4’ and ‘3’). This is an important point to absorb, in terms of one’s personal understanding, and you should recall the (almost interminable!) struggle over that very point in class discussions.

**Note.** If $r(k-1, l)$ and $r(k, l-1)$ are both even, it can be argued that a little more is true, namely, $r(k, l) < r(k-1, l) + r(k, l-1)$. Thus it follows (e.g.) that $r(4, 3) \leq 9$, and – in fact – $r(4, 3) = 9$, as is shown by exhibiting a ‘Ramsey graph’ having 8 vertices (namely one with no 4-clique and no 3-independent set).

**Which Ramsey numbers are known?** Very few non-trivial (meaning, of course, that $k, l \geq 3$) Ramsey numbers are known, despite huge efforts at determining them.

However one can at least say something (using the Erdős-Szekeres theorem) about how big they are, at most. For example one may easily (though crudely) argue that $r(5, 5)$ is at most 70. How? Simply by making a succession of applications of the E-S theorem:

- $r(4, 3) \leq r(3, 3) + r(4, 2) = 6 + 4 = 10.$
- $r(4, 4) \leq r(3, 4) + r(4, 3) \leq 10 + 10 = 20.$
- $r(5, 4) \leq r(4, 4) + r(5, 3)$ [see separate calculation] \leq 20 + 15 = 35. 
  \[ r(5, 3) \leq r(4, 3) + r(5, 2) \leq 10 + 5 = 15 \]
- $r(5, 4) \leq 35.$
- $r(5, 5) \leq r(4, 5) + r(5, 4) \leq 35 + 35 = 70.$
- $r(5, 5) \leq 70.$

**Comment.** Of course one may easily improve the last inequality by using actual known results for $r(4, 3), r(4, 4)$ and $r(5, 3)$: $r(4, 3) = 9, r(4, 4) = 18, r(5, 3) = 14$. In fact, it happens to be known that $43 \leq r(5, 5) \leq 49$.

**For the record.** At the time of writing, here are the only Ramsey numbers (the 2-variable ones, that is, and with $k, l \geq 3$) whose values are known:

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13 A good web reference is [http://mathworld.wolfram.com/RamseyNumber.html](http://mathworld.wolfram.com/RamseyNumber.html)
Of the more general (non-trivial) Ramsey numbers, only one is known!! It is $r(3, 3, 3)$, whose value is known to be 17. The meaning of that is that if one chooses any 17 (or more) points and joins every two of them using any one of 3 colours, red, green, and blue (say; by the way, that ‘$3$’(colours) is the number of variables – coordinates – and not the ‘3’ in that ‘3, 3, 3’), then, however one does it, there will always result either a red triangle, a green triangle, or a blue triangle (that’s what the ‘3, 3, 3’ is about). Recall that an interpretation of $r(3, 3)$ being 6 is that if one chooses any 6 (or more) points, and joins every two of them using 2 colours, red and blue (say), then, however one does it, there will always result either a red triangle or a blue triangle.

You should recall, too, how that may returned into a game (which you could play with your friends, introduce to children, use your imagination…) for 2 people who have paper and two differently coloured crayons. On the paper mark 6 points, and then play. A ‘move’ is to join two points by an edge (it doesn’t have to be ‘straight’). Players move alternately, using their colour. Who ‘loses’? The first one to complete a ‘triangle’ (a 3-clique, call it what you will). By Ramsey theory there must be a loser\textsuperscript{14}.

It would, of course, be more interesting to play with 18 points!! There, the loser is the one who first completes a 4-clique:

\begin{center}
\includegraphics[width=0.5\textwidth]{triangle.png}
\end{center}

Once again, by Ramsey theory, there must be a loser.

\textsuperscript{14} Must the player who moves first be the loser, providing the second player plays appropriately?